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Closed Form Effective Conformal Anomaly Actions in D≥4

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Abstract

We present closed-form coordinate invariant effective actions for both types A and B of Weyl anomalies in all dimensions D≥4. Their nonlocalities reflect (as they must) the anomalies' underlying loop diagram origins. In particular, despite contrary appearances, the properly interpreted Riegert–Polyakov type A actions do yield the anomalies and reproduce the lowest order pole structure. For type B, where no correct candidate existed, our action both varies correctly and reflects the proper cutoff – and logarithmic scale – dependence. It is constructed in terms of novel Weyl invariant tensor operators.

1 Introduction

The nature of conformal or Weyl anomalies beyond the (unique) Polyakov action at D=2 level is reasonably well established. In particular, we know [1] that there exist two families, with very distinct origins (in terms of the IR and UV parts of the underlying matter closed loops) and characteristics (in terms of their own Weyl variations) in arbitrary even dimension D=$2n$ (there are no Weyl anomalies in odd D). There must exist effective actions, whose Weyl variations yield the respective anomalies expressed in terms of a background gravitational field, that keep track of the effects of integrating out the matter closed loops. The actions should be covariantizations of lowest order expressions $I_0[g] \sim \int \sqrt{-g}\, g^{\mu\nu} < T_{\mu\nu}..T_{\alpha\beta} > g^{\alpha\beta}$ and are necessarily nonlocal (any local parts are irrelevant) but their Weyl variations under

$$\delta g_{\mu\nu} = 2\phi(x) g_{\mu\nu}\ , \quad \delta\sqrt{-g} = D\phi\sqrt{-g} \tag{1}$$

yield local anomalies $\mathcal{A}(x) \equiv \delta I[g_{\mu\nu}]/\delta\phi(x)$. The built-in integrability condition on second variations,

$$\delta\mathcal{A}(x)/\delta\phi(x') = \delta^2 I/\delta\phi(x')\delta\phi(x) = \delta\mathcal{A}(x')/\delta\phi(x) \tag{2}$$

serves as a useful check on candidate $\mathcal{A}$'s and on form of compensator field actions.

The origin of the anomalies in closed loop graphs imposes obvious constraints on the actions' momentum dependence. These seem to clash with the formally attractive diffeomorphism-invariant D≥4 generalizations of the D=2 action for type A, and make type B actions (which start at D=4) hard to define at all. Let us briefly recall the minimal requirements on effective actions in the present context. First, they must obviously vary into the local anomalies $\mathcal{A}(x)$; this also dictates their momentum dimension and nonlocality structure. Secondly, loss of classical conformal invariance having been accepted as the price for preserving coordinate invariance, they must be invariants. On the other hand, they need not be unique, *i.e.*, there can and (as we will show) do, exist nonlocal Weyl invariants. Nor do they have to be obtainable (unlike in D=2) by integrating out an action

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for a compensating field that is physically acceptable (*e.g.*, ghost-free); indeed we will see this is unlikely in D>2.

In this paper, I present complete, closed form actions satisfying the above requirements for both types A and B in all D, most explicitly in D=4, as well as a Weyl-invariant nonlocal action to demonstrate non-uniqueness. For type A the correct action is that proposed by Riegert [2], as an extension of the D=2 Polyakov form to D=4. While it seemed to yield the anomaly (though its form must be slightly improved), it also seemingly violated a single pole ($\Box^{-1}$) nonlocality required by dimension, having instead an explicit $\Box^{-2}$ factor. This dimensionality paradox disappears when two facts are taken into account: first, the demand that diffeomorphism invariance be manifest *i.e.*, that everything has to be expressed in terms of curvature (to lowest order expansion of the action about flat space) is *not* mandatory and second, the single pole requirement only constrains the lowest order part; suitable use of the first freedom will indeed turn $\Box^{-2}$ into $\Box^{-1}$ there. At higher orders, higher poles can and indeed should be present. For type B, a Riegert-like expression (the only extant candidate) is not only of wrong dimension but does not, in fact, vary correctly. Instead, I provide proper all-order effective actions through use of new operators, Weyl-invariant when acting on the Weyl tensor, that are compatible with the physics.

2 Type A: D=2 revisited

In D=2, where type A is the only possible anomaly, things are as usual very simple; they are also very (sometimes too!) suggestive for generalizing to D=4 and beyond. By power counting alone, the anomaly $\mathcal{A}(x)$ must have dimension 2; the only local diffeo-invariant is the Euler density $\mathcal{E}_2(x)$,

$$\mathcal{A}(x) = \sqrt{-g}\, R(x) \equiv \mathcal{E}_2(x),\ \delta\mathcal{E}_2(x) \equiv 2\sqrt{-g}\,\Box\phi \tag{3}$$

whose Weyl variation is also indicated (overall constant coefficients are omitted for brevity). From (2), integrability of $\mathcal{A}$ is manifest:

$$\delta\mathcal{E}_2(x)/\delta\phi(x') \equiv 2\sqrt{-g}\,\Box\delta^2(x-x') \equiv \delta\mathcal{E}_2(x')/\delta\phi(x) \tag{4}$$

Note also that the scalar density operator $\Delta_2 \equiv \sqrt{-g}\,\Box$, when – and only when – acting on a scalar is itself Weyl invariant at D=2,

$$\delta\Delta_2 \equiv \delta[\partial_\mu(\sqrt{-g}\, g^{\mu\nu})\partial_\nu] = 0\ . \tag{5}$$

Together, the behavior (2,4) of $\mathcal{E}_2$ and Δ_2 under variation mean that

$$\delta(\mathcal{E}_2/\Delta_2) = 2\phi(x) \tag{6}$$

i.e., that this nonlocal scalar operator acts like a Weyl compensator field. This is of course what underlies the Polyakov construction,

$$I_2 = \tfrac{1}{4}\int d^2x\mathcal{E}_2\Delta_2^{-1}\mathcal{E}_2\ , \quad \delta I_2/\delta\phi(x) = \mathcal{E}_2(x)\ . \tag{7a}$$

Strictly speaking, use of (5) in varying (7a) is not entirely correct as Δ_2 must act on a scalar, rather than on a density like $\mathcal{E}_2$ to be invariant; this is easily remedied here, by writing (7a) as

$$I_2 = \tfrac{1}{4}\int \sqrt{-g}\,(\mathcal{E}_2\Delta_2^{-1})(\mathcal{E}_2/\sqrt{-g}\,)\ , \tag{7b}$$

whose variation is more easily verified (see below) than is (7a) to yield

$$\delta I_2/\delta\phi(x) = \mathcal{E}_2(x) \ . \tag{7c}$$

This seeming pedanticism will be better appreciated in D≥4. The action (7) is necessarily nonlocal, so that I_2 cannot be absorbed as a local counterterm might. The pole behavior of the action is clearly $\sim p^{-2}$, in accord with the power counting of the 2-point closed loop $\sim (\int d^2p/p^4)R_L^2$ where $R_L \sim (pph)$ is the linearized scalar curvature in an expansion about flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The underlying correlator is $< T^{\mu\nu}(p)T^{\alpha\beta}(-p) >$, here multiplied by $h_{\mu\nu}h_{\alpha\beta}$ and keeping track of the four factors of momentum in the $< TT >$ numerator. However, this counting is true only to leading, here quadratic, order in $h_{\mu\nu}$. If we expand (7), we see that all "dressings" of the curvatures in powers of h, keeping the flat space Δ_2^{-1}, maintain the p^{-2} overall behavior, as they should diagrammatically since this is still a 2-point function. Instead, even in D=2, if we also expand Δ_2^{-1}, about flat space

$$\begin{aligned} \Delta_2^{-1} &\equiv [\Box_0 + \partial_\mu \mathcal{H}^{\mu\nu}\partial_\nu]^{-1} = \Box_0^{-1}(1 - \Box_0^{-1}\partial_\mu\mathcal{H}^{\mu\nu}\partial_\nu + ...), \qquad \Box_0 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \\ \mathcal{H}^{\mu\nu} &\equiv \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu} = -(h^{\mu\nu} - \tfrac{1}{2}\,\eta^{\mu\nu}h) + ... \end{aligned} \tag{8}$$

there are increasing powers of p^{-2}, in total agreement with the diagrammatics: a 3-point closed loop generically acquires another p^{-2} from the extra propagator and so on for the n-point expansion. Indeed, although it seems to be folklore that anomalies must have only a $\Box_0^{-1}$ nonlocality, this is only the case for the leading term. This fact will be essential in D=4. Despite these higher poles, the D=2 action is derivable by integrating out a perfectly physical ghostfree compensator field's action, $I_2[\sigma] = \int d^2x[\frac{1}{2}\sigma\Delta_2\sigma + \sigma\mathcal{E}_2]$; this will no longer hold in higher dimension.

3 Type A, D≥4

As shown in [1] there is a unique generalization of the D=2 anomaly to any D=2n, namely the same "infrared" type is given by the Euler density at D=2n,

$$\mathcal{A}_{2n} = \mathcal{E}_{2n} \equiv (-g)^{1/2}\epsilon^{1..2n}\epsilon^{1'..2n'}R_1...R_n \ , \tag{9}$$

where $\mathcal{E}_{2n}$ vanishes identically in lower D (since the Levi–Civita symbol does). Then integrability is always satisfied because $\mathcal{E}_{2n}$ Weyl transforms as

$$\delta\mathcal{E}_{2n}(x) \equiv \mathcal{G}_{2n}^{\mu\nu}D_\mu D_\nu\phi(x) \tag{10}$$

where $\mathcal{G}_{2n}^{\mu\nu}$ is an identically conserved tensor (as it must be, since $\mathcal{E}_{2n}$ and its variations are total divergences). For concreteness we will work specifically in D=4, then indicate the generalization to arbitrary D. Here, $\mathcal{G}_4^{\mu\nu}$ is of course the Einstein tensor (that indeed vanishes at D=2), hence it follows directly from (9) that

$$\delta\mathcal{E}_{2n}(x)/\delta\phi(x') = \mathcal{G}_{2n}^{\mu\nu}(x)D_\mu D_\nu\delta(x-x') = \delta\,\mathcal{E}_{2n}(x')/\delta\phi(x) \ . \tag{11}$$

It is therefore tempting to write the same action as in D=2, in terms of the suitable generalization Δ_4 of Δ_2. Just by constant scale invariance, this must start as $\Delta_4 \sim \sqrt{-g}(\Box^2 + ..)$. Indeed, the precise form of Δ_4 (acting on a scalar) was discovered by Paneitz [3] and itself generalizes to arbitrary $D = 2n$:

$$\Delta_4 = \sqrt{-g}(\Box^2 + 2D_\mu(R^{\mu\nu} - \tfrac{1}{3}\, g^{\mu\nu} R)D_\nu)\ ; \tag{12}$$

it is self-adjoint and reduces to $\Box_0^2$ at flat space. So it appears as if the natural extension of the Polyakov action due to Riegert [2] is correct. Actually, $\bar{\mathcal{E}}_4 \equiv \mathcal{E}_4 + \frac{2}{3}\sqrt{-g}\,\Box R$ rather than $\mathcal{E}_4$ varies as in D=2

$$\delta\bar{\mathcal{E}}_4 = \Delta_4\phi\ , \quad \delta\Delta_4 = 0 \quad \delta(\bar{\mathcal{E}}_4/\Delta_4) = \phi\ , \tag{13}$$

the last two equations being valid only when Δ_4 acts on scalars. Hence the desired extension of (7) is

$$I_4^A = \int d^4x\, \sqrt{-g}\, \bar{\mathcal{E}}_4 \Delta_4^{-1}(\bar{\mathcal{E}}_4/\sqrt{-g}\,)\ , \quad \delta I_4^A/\delta\phi = \bar{\mathcal{E}}_4 \tag{14}$$

where we have rewritten the action of [2], as we did for D=2 in (7b) to make Δ_4^{-1} act on the scalar $(\bar{\mathcal{E}}_4/\sqrt{-g})$. It is not quite obvious yet that (14) with its funny $\sqrt{-g}$'s does vary correctly, but, as we show below, it does. If we recall that $\Box R$ itself derives from a local (and hence irrelevant, removable) term, $\delta\frac{1}{4}\int d^4x\, R^2\sqrt{-g}\,/\delta\phi = \sqrt{-g}\,\Box R$, we see that since (14) varies into $\bar{\mathcal{E}}_4$, it effectively also varies into $\mathcal{E}_4$.

In [1] and elsewhere, (14) was criticized on the seemingly correct grounds that (through Δ_4^{-1}) it had a $\Box_0^{-2}$ pole, incompatible with the 3-point function $\int hhh\langle TTT\rangle$, $\sim \int \frac{d^4p}{(p^2)^3}\,(R^3) \sim \int R^3/\Box_0$ just by momentum counting around loops with three external curvatures, *i.e.*, the leading $\mathcal{O}(h^3_{\mu\nu})$ term had to be $\Box_0^{-1}$, and not $\Box_0^{-2}$, nonlocal. The dimensional argument is correct, but the objection was based on insisting that the $\mathcal{O}(h^3_{\mu\nu})$ part of the action be manifest linearized gauge invariant: everything had to be stated in terms of curvatures. To understand the resulting paradox, first simply expand (14) in $h_{\mu\nu}$:

$$I_4^A[h^3] = \int d^4x(\mathcal{E}_4 + \tfrac{2}{3}\,\Box_0 R)\Box_0^{-2} \times (1 - \Box_0^{-2}\partial_\mu(R^{\mu\nu} - \tfrac{1}{3}\,\eta^{\mu\nu}\, R)\partial_\mu + ...)(\mathcal{E}_4 + \tfrac{2}{3}\Box_0 R)\ . \tag{15}$$

Here all curvatures are needed only to their linearized $\mathcal{O}(h)$ order and all derivatives are also flat space ones; all corrections to those quantities either lead to $\mathcal{O}(h^4)$ or are $\mathcal{O}(h^3)$ but harmless, $\sim \Box_0^{-1}$. Next, get rid of the unity part of the Δ_4^{-1} expansion: the quadratic terms are the local $\int d^4x\, R^2$, the cubics are $\sim \int d^4x[\mathcal{E}_4\Box_0^{-1}R + \Box_1 R\Box_0^{-1}R]$ where $\Box_1$ is the $\mathcal{O}(h)$ part of $\Box$; they are single-pole. Now pass to the correction term which seems to have a $\Box_0^{-4}$. However, being linear, it only multiplies the quadratic $(\Box_0 R)\Box_0^{-2}\Box_0 R \sim R^2$, so it is $\Box_0^{-2}$ at worst. Before proceeding further, note two important features of cubic integrals: first, the position of $\Box_0^{-1}$ among the 3 factors is irrelevant; second, integration by parts rules are very useful, *e.g.*, (for any S) $\int d^4x\, S\,\partial_\mu R\partial^\mu R = -\frac{1}{2}\int d^4x\, S\Box_0 R^2$. Both are used implicitly below. The dangerous cubic terms in (15) are then of the form

$$\int d^4x\, R\,\Box_0^{-2}\partial_\mu(R^{\mu\nu} - \tfrac{1}{3}\,\eta^{\mu\nu} R)\partial_\nu\, R\ . \tag{16}$$

The pure R^3 part, $\sim \int R_{,\mu}R_{,\mu}\Box_0^{-2}R = -\frac{1}{2}\int R^3/\Box_0$ is obviously safe. This leaves the first, $R^{\mu\nu}$-dependent one,

$$\int d^4x R\Box_0^{-2}\partial_\mu(R^{\mu\nu}\partial_\nu)R = -\int d^4x R_{,\mu}R_{,\nu}\Box_0^{-2}R^{\mu\nu}, \tag{17}$$

which is certainly $\sim \Box_0^{-2}$ as it stands. Note that there is no dimensional contradiction: the extra $\Box_0^{-1}$ is compensated for by the extra $\partial_\mu\partial_\nu$ in the numerator, but these are not mutable into a $\Box_0$

by parts integration as long as we write everything in terms of curvatures alone. This seeming impasse disappears by relaxing the latter requirement and expressing the Ricci tensor in terms of its metric definition,

$$2R_{\mu\nu} = \Box_0 h_{\mu\nu} - (\partial^2_{\mu\alpha} h^\alpha_\nu \ + (\nu\mu)) + h^\alpha_{\alpha,\mu\nu} \ . \tag{18}$$

The $\Box_0 h_{\mu\nu}$ term is manifestly $\sim \Box_0^{-1}$; the remaining ones also provide an additional $\Box_0$, after integration by parts. The result of a simple calculation yields the equality

$$\int d^4x \, R_{,\mu} R_{,\nu} \Box_0^{-2} R^{\mu\nu} = \tfrac{1}{2} \int d^4x \left[R_{,\mu} R_{,\nu} h^{\mu\nu}/\Box_0 - \tfrac{1}{4} R^2 h^\alpha_\alpha + \tfrac{1}{2} R^3/\Box_0 \right] \ . \tag{19}$$

whose right side, although its first term is irreducible to "curvatures/$\Box_0$" is of course just as gauge-invariant under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, $\delta R = 0$ as the left.

This demonstrates the single pole nature of Riegert's action to $\mathcal{O}(h^3)$, which is all that can be demanded of it. Thereafter, there will clearly appear higher and higher poles in the $h_{\mu\nu}$-expansion of Δ_4^{-1}. Indeed, as explained previously, each successive additional vertex insertion into the loop diagram involves an extra propagator and so, generically an (acceptable) extra power of $\Box_0^{-1}$. For higher D, the Δ_{2n} will go as $\Box^n$, but again the leading, $(n+1)$-point function must go as $\int (d^{2n}p)/(p^2)^{n+1} \sim p^{-2}$, and it will, by similar considerations as for D=4, with

$$I_A = \int d^{2n}x \, \bar{\mathcal{E}}_{2n} \Delta_{2n}^{-1} \, \bar{\mathcal{E}}_{2n} \tag{20}$$

and $\Delta_{2n} \sim \Box^n + ..$, $\bar{\mathcal{E}}_{2n} \sim \mathcal{E}_{2n} + ...$ where the additional terms in Δ_n and $\bar{\mathcal{E}}_{2n}$ are of lower/higher derivative order respectively, and with $\delta(\bar{\mathcal{E}}/\Delta) = \phi$ as in D=2,4.

To keep these results in perspective, one must remember that existence of an action which varies correctly is almost tautological since its leading, $\mathcal{O}(h^3)$, part is obtained (in D=4, say) from the 3-point loop $\langle TTT \rangle$ by contracting it with three metrics; it thereby also has the correct derivative dimensions. Hence any action that varies correctly into the anomaly will have the right dimensions as well. The new considerations here really make explicit first that (7) (as we will see from its second incarnation (7b)) indeed varies into the anomaly and second that it does embody (albeit in hidden form) the lowest order (and permitted higher order) pole behavior dictated by those dimensional properties. It is also clear that while (7) or (20) can be obtained by integrating out local QFT actions, the latter necessarily have ghost field kinematics for D>2:

$$I_{cov}[\sigma] = \int d^{2n}x [\tfrac{1}{2} \, \sigma \Delta_{2n} \sigma + \sigma \bar{\mathcal{E}}_{2n}] \ , \tag{21a}$$

since Δ_{2n} here really does start as $\Box_0^n$. Even the leading, $\mathcal{O}(h^3)$, $1/\Box_0$ terms cannot be obtained from a nonghost scalar action for D>2, simply by dimensions. Consider for D=4, the general form

$$I_3[\sigma] = \int d^4x [\tfrac{1}{2} \, \sigma(\Box_0 + \Box_1 + aR)\sigma + \sigma Z] \ , \tag{21b}$$

Here the scalar Z would obviously have to be of derivative dimension 3; quite apart from any details, no scalar Z (nor a vector action conserved source Z^μ) can embody this. One could introduce $Z \sim R^{3/2}$ terms, but even with this nonpolynomial ansatz it does not seem feasible to get the desired $\mathcal{O}(h^3)$ action. [As mentioned, the "natural" compensator field action contains [2, 6] $(\partial\sigma)^4$ terms.]

4 Type B, D≥4

The type B anomaly, which was in fact the first one discovered in D=4 [5], has no D=2 antecedents at all. Its hallmarks are that the anomaly itself is Weyl invariant,

$$\delta \mathcal{A}_B(x)/\delta\phi(x') \equiv 0 \tag{22}$$

and that the action depends on the UV cutoff in a logarithmic way. There is only one $\mathcal{A}_B$ in D=4, but their number rises rapidly with dimension, *e.g.*, there are 3 varieties at D=6. For D=4,

$$\mathcal{A}_B \equiv \sqrt{-g}\, tr\, C^2 \ , \quad \delta\mathcal{A}_B/\delta\phi \equiv 0 \ ; \tag{23}$$

the invariance of $\mathcal{A}_B$ follows from the fact that $C^{\mu}{}_{\nu\alpha\beta}$ is the conformal-invariant "index location" of the Weyl tensor. The required logarithmic behavior was first approximated in [5] by the lowest order action

$$I_B^4 \approx \int d^4x \sqrt{-g}\, tr\, C \ln(\Box/\lambda^2) C + \mathcal{O}(h^4) \tag{24}$$

with $\Box$ taken to $\mathcal{O}(h)$. From (8), this is plainly not quite correct, but certainly reproduces some of the characteristics, including the dimensional cutoff dependence. What we really want of course is an argument $\tilde{\Delta}$ in the logarithm that varies as $\delta\tilde{\Delta}/\tilde{\Delta} = \phi$, while being simultaneously a scalar (densities cannot be argument of logs by covariance) and of dimension 4, to bring in a λ^{-4}. What is the appropriate "compensator" $\tilde{\Delta}$? One's immediate reaction is that $\tilde{\Delta} = \Delta_4/\sqrt{-g}$, since $\ln(\Delta_4/\sqrt{-g}\,\lambda^4)$ fulfills all requirements, but again only when acting on a scalar. [Indeed the role of $\tilde{\Delta}$ is merely to "legally" bring in ln $\sqrt{-g}$ as the compensator, as well as display the need for a cutoff!] However, to keep Δ_4 acting on scalars only would mean writing

$$\tilde{I}_B^4 \sim \int d^4x \sqrt{-g}\, \ln(\Delta_4/\sqrt{-g}\,\lambda^4) tr\, C^2 \ , \tag{25}$$

which seems superficially to vary correctly; unfortunately, $\tilde{I}_B^4$ is really a total divergence, with vanishing variation.

Let us dispose of one other candidate action, based on the type A construction, that appears to yield $\mathcal{A}_B$ exactly except for the fact that it does not reflect the ln "$\Box^4/\lambda^4$" behavior required in type B *i.e.*, has wrong dimension. It is an instructive lesson, therefore, that it also thereby does *not* Weyl-vary correctly, best seen through the fine point that distinguished (7a) from (7b) earlier: The type A action is based on the fact that $\bar{\mathcal{E}}/\Delta$ varies into ϕ itself, so one might think that an action of the form $\bar{I}_B^4 \sim \int d^4x (\bar{\mathcal{E}}_4/\Delta_4)\sqrt{-g}\, C^2$ is the most economical way to achieve $\delta\bar{I}/\delta\phi = \sqrt{-g}\, C^2$. However, we must take into account the requirement that Δ_4 act on a scalar in order to be invariant, here on C^2 rather than on the density $(\sqrt{-g}C^2)$, rewriting $\bar{I}_B^4 = \int d^4x \sqrt{-g}(\bar{\mathcal{E}}_4/\Delta_4)(\sqrt{-g}C^2/\sqrt{-g}\,)$. But in this form there are two extra contributions to the variation, one each from the left $\sqrt{-g}$ and the right $1/\sqrt{-g}$:

$$\delta\bar{I}_B^4/\delta\phi = \sqrt{-g}\, C^2 + (\bar{\mathcal{E}}_4 \Delta_4^{-1} C^2 - C^2 \Delta_4^{-1} \bar{\mathcal{E}}_4), \tag{26}$$

which do *not* vanish, unlike for type A, where we had the symmetric $\bar{\mathcal{E}}_4 \Delta_4^{-1} \bar{\mathcal{E}}_4$ and hence the corresponding extra term, $(\bar{\mathcal{E}}\Delta^{-1}\bar{\mathcal{E}} - \bar{\mathcal{E}}\Delta^{-1}\bar{\mathcal{E}})$, vanishes when varying (20). The above "wrong" construction, however, is useful in exhibiting the existence of a nonlocal Weyl invariant, and hence the formal non-uniqueness, but beyond leading $h_{\mu\nu}$-order, of our effective actions: It is easy to verify the Weyl invariance of

$$I_{INV} = \int d^4x \sqrt{-g}\, (\sqrt{-g}\, C^2) \Delta_4^{-1} C^2 \ ; \tag{27}$$

the second term in the equivalent of (26) vanishes as both factors are identical. More generally, nonlocal conformal (and diffeo-)invariants can only begin at next-lowest, $\mathcal{O}(h^{n+2})$, in D=$2n$. In this sense our explicit covariant actions, which start at $\mathcal{O}(h^{n+1})$, are unique: any others differ from them by invariants that begin beyond the leading $\mathcal{O}(h^{n+1})$ terms.

Finally let us turn to the desired all-order I^4_B. What makes it expressible in closed form is the existence [6] of a covariant 4$^{\text{th}}$ derivative order tensor density operator Δ^c_4, any power of which, when acting on a 4-tensor (but non-density!) with the algebraic properties of the Weyl tensor, remains invariant: $\delta(\Delta^c_4) \equiv 0$, and therefore also

$$\delta\left\{(\Delta^c_4)^{\mu\,\nu'\,\alpha'\,\beta'}_{\mu'\,\nu\,\alpha\,\beta}\;C^{\mu'}{}_{\nu'\alpha'\beta'}\right\} \equiv 0\ . \tag{28}$$

Furthermore $(\Delta^c_4 C)$ itself retains the algebraic properties of C. The detailed form of this operator is complicated but we only need to know it exists; and in terms of it the action is simple:

$$I^4_B = -\tfrac{1}{4}\int d^4x\sqrt{-g}\,C_\mu{}^{\nu\alpha\beta}[\ln(\Delta^c_4/\sqrt{-g}\lambda^4)C]^\mu_{\nu\alpha\beta}\ . \tag{29}$$

The only non-vanishing variation in (29) is entirely due to varying the $\ln\sqrt{-g}$ factor,

$$\delta I^4_B \sim \tfrac{1}{4}\int d^4x\sqrt{-g}\,C^2\delta\ln\sqrt{-g} = \int d^4x(\sqrt{-g}C^2)\delta\phi \tag{30a}$$

all the rest being obviously invariant; in particular the left factor $(\sqrt{-g}\,g^{\cdot\cdot}g^{\cdot\cdot}C^{\cdot}\ldots)$ is. [In more detail, it is the density Δ^c_4 that is Weyl invariant, just because it contains 4 derivatives, like Δ_4. Hence any power in the log's expansion, $(\Delta^c/\sqrt{-g}\,)^m C$, means $(\frac{1}{\sqrt{-g}}\,\Delta^c\ldots\frac{1}{\sqrt{-g}}\Delta^c)C$; the expansion correctly avoids having Δ^c act on any density and only the last $1/\sqrt{-g}$ factor contributes.] We conclude then, that I^4_B generates the anomaly:

$$\delta I^4_B/\delta\phi = \sqrt{-g}\,C^2 \tag{30b}$$

The various possible $\mathcal{A}_B(x)$ in D>4 will clearly be expressible in terms of the corresponding Δ^c_{2n}, which are also sure to exist:

$$I^{2n}_B \sim \int d^{2n}x\sqrt{-g}\,Z_\mu{}^{\nu\alpha\beta}\left[\ln\left(\frac{\Delta^c_{2n}}{\sqrt{-g}\,\lambda^{2n}}\right)C\right]^\mu_{\nu\alpha\beta} \tag{31}$$

where Z is the “rest” of the local invariant in question, *e.g.*, $Z \sim (C_1..C_{n-1})$.

5 Discussion

We have provided diffeomorphism invariant effective actions for both type A and B anomalies in closed form at all D≥4, and shown that they were also in accord with their underlying loop origins. For type A, this meant reconsideration of the Riegert action [2] to show that – despite initial appearances – it both varied correctly (when the proper density factors were inserted) and (as had to be) respected the single pole requirement to (and only to) lowest order about flat space when expanded in metric form. For type B, the existence of Weyl invariant tensor density operators Δ^c_{2n} acting on tensors with the algebraic properties of the Weyl tensor (just like the simpler Δ_{2n} were Weyl invariant when acting on scalars) enabled us to provide closed form actions compatible with the $\log(\Box/\lambda^2)^n$ behavior dictated by the diagrams; candidate actions behaving differently varied unacceptably. Construction of a nonlocal Weyl-invariant, starting at $\mathcal{O}(h^{n+2})$, showed our actions' nonuniqueness, at least without imposing further constraints. However, despite their closed form nature, the physical meaning of these actions is still not sufficiently clear. For type A especially, we saw that in contrast to the D=2 case, there is no physically acceptable local action which yields ours upon integrating out a compensating field. This seems to be true not merely of the full covariant forms but already for the unique $\mathcal{O}(h^{n+1})$, $\Box_0^{-1}$, leading terms near flat space. Needless to say, we know still less about any such underlying basis for the type B action, with its logarithmic form. Given the continuing interest in conformal anomalies (for very recent very examples see e.g., [7, 8]), a better understanding seems worth pursuing.

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